J. Fluid Mech. (1970), *vol.*, 41 *part* 4, *pp.* 751–758
Printed in Great Britain

On bubbles rising in line at large Reynolds numbers

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(Received 24 February 1969 and in revised form 16 October 1969)

Conditions for two gas bubbles in a liquid to rise steadily in a vertical line are derived theoretically with these assumptions: large Reynolds number, no surface contamination, spherical shape, negligible gas density and viscosity. Drag coefficients are found, and are lower than for single bubbles. The bubbles have equilibrium distances apart, which are calculated to a first approximation. The equilibrium is shown to be stable to small vertical disturbances but unstable to horizontal ones. Similar results exist for lines of more than two bubbles, but are not calculated in detail.

1. Introduction

In certain liquids, a bubble of gas can rise steadily while remaining very nearly spherical, at a velocity U such that the Reynolds number

$$R = U\,d\rho/\mu \tag{1.1}$$

is of the order of a few hundred. Here d is the bubble diameter, and ρ and μ are the density and viscosity of the liquid. If the liquid is exceptionally pure, experimental results for the drag coefficient agree well with calculations from boundary-layer theory (Moore 1963), in which the density and viscosity of the gas in the bubble are neglected.

This paper describes an extension of Moore's theory to a number of spherical bubbles of the same size rising in a straight vertical line in an unbounded fluid. Lines of bubbles commonly appear in drinks containing carbon dioxide (e.g. beer), where particular points on the surface of the glass nucleate many bubbles in succession. They also occur in experiments where gas is blown through a tube whose end is fixed below the surface of a liquid. Regrettably, the theory is inapplicable to the first case, both because beverages contain surface-active solutes (consider the 'head' on a glass of beer), and because their bubbles usually grow larger while rising as they absorb more gas. Unless they are dissolving, bubbles must grow in any liquid, because the pressure on them decreases as they rise. However, this growth is slight if the gas is insoluble and distance travelled is very much smaller than the height of a barometer containing the liquid. For simplicity, we shall ignore the growth. As a result, bubbles must approach a constant terminal velocity, and we can neglect variations with time of the flow relative to the bubbles.

The calculation proceeds as follows. Assuming a first approximation to the flow pattern to be irrotational, as if the fluid were inviscid (see Moore 1963, Batchelor 1967), we find the velocity distribution around a bubble in §2, and show that any two bubbles in the line are repelled by a force varying inversely as the fourth power of the distance between them, to a good approximation. In §3 we obtain the boundary-layer corrections to that velocity distribution, allowing for the fact that each bubble after the first in the line passes through fluid which has been in the boundary layers of the previous bubbles. This is done in the same way that Harper & Moore (1968) treated the circulatory flow inside a drop, where fluid entering the boundary layer has also passed through one before. (This part of the work is very much easier than Harper & Moore's, because there is a simple solution in closed form for the velocity perturbations, and no difficulties occur at the stagnation points because the viscosity of the gas is ignored.) The viscous drag on a bubble is obtained to order $R^{-\frac{3}{2}}$, and turns out to decrease monotonically down the line. However, Harper & Moore's method can only be used without modification if the line is much shorter than $dR^{\frac{1}{2}}$, for diffusion of vorticity in the wakes between bubbles is not taken into account. The present theory therefore cannot give good numerical results for more than two bubbles, although the viscous drag must always decrease down the line. Distortion from a spherical shape also will not alter that property of the drag, because each bubble moves upwards through fluid which has been given an upward velocity by its predecessors.

For a constant speed U, each bubble must have the same total drag coefficient. The configuration is stable to vertical displacements, because upward movement of any bubble relative to the others would not alter the viscous drag, but would increase the repulsion from bubbles above and decrease it from bubbles below, thereby forcing the displaced bubble downwards. Similarly, any downwards movement would be halted and reversed. The velocity U can be found as a function of diameter in the usual way, by setting the resultant force due to the fluid motion equal to the buoyancy force $\frac{1}{6}\pi d^3 \rho g$. As this is the only part of the problem in which gravity appears if distortion from spherical shape is neglected, we can simplify the equations of motion by subtracting the hydrostatic term from the pressure, and using the modified pressure (Batchelor 1967).

We show in §4 how the wake vorticity from the first bubble is amplified and distorted when the second is slightly out of line. A longitudinal trailing vortex system appears below the second bubble, revealing the existence of a lift force on it which pushes it even further out of line. The equilibrium position directly under the first bubble is therefore unstable in a pure liquid.

Hawthorne & Martin (1955) described vortices arising in a similar way around a hemisphere in a boundary layer, but they neglected viscous diffusion of vorticity near its surface. So did Lighthill (1956, 1957a, b), who gave the inviscid theory in much greater detail than is needed here. Section 4 uses different variables and co-ordinates, chosen to simplify the calculations with viscosity included.

It seems possible that in a liquid containing a surface-active solute the system may become stabilized, for the following reason: as the first bubble rises, adsorbed solute is carried around on its surface from front (top) to rear (bottom), and so the front of the bubble has a lower concentration of solute and hence higher surface tension than the rear. (This slows down the motion; see Levich 1962.) The fluid near the rear stagnation point, with its low surface tension, is then carried down the middle of the wake, and is surrounded by fluid with higher values of surface tension. If the second bubble is now slightly displaced from a central position, say to the left, the surface tension will be higher on its left side than its right, and there will be a rightwards force on it tending to pull it back. The calculation is involved, however, and will be described elsewhere.

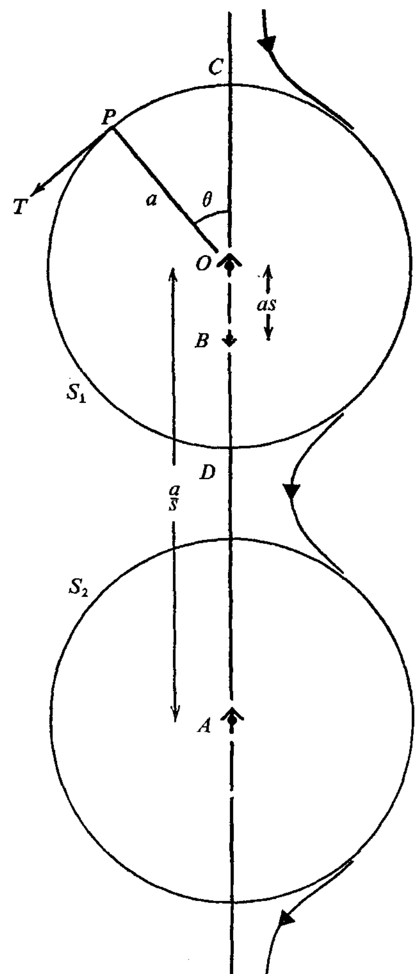

FIGURE 1. Two bubbles rising in the same line, with $s = 0{\cdot}4$. The image dipoles shown at O, A, B give the velocity around S_1 to order Us^5. One streamline of the flow relative to the bubbles is shown, but not where it would pass very close to their surfaces.

2. The irrotational first approximation to the flow

We begin by considering two spherical bubbles only, and then extend the result to a greater number. Let the two bubbles, S_1 and S_2 in figure 1, whose centres are at O and A, be at rest in a stream with steady uniform velocity U

at infinity in the direction OA. Define a separation parameter s (which is assumed small) by the equation

$$s = \frac{a}{OA}, \tag{2.1}$$

where a is the radius of either sphere. We use right-handed spherical polar co-ordinate systems (r_n, θ_n, χ_n) centred on the nth bubble from the upstream end, with $\theta_n = 0$ pointing upstream, and χ_n the azimuthal angle increasing anti-clockwise as seen from above.

If the flow past the spheres is taken to be irrotational, it can be found by superposing the uniform stream U and a set of image doublets (Basset 1961). The resultant velocity $\bar{q}_\theta$ at P is in the direction PT and given by

$$\bar{q}_\theta = \tfrac{3}{2}U \sin\theta_1\{1 - s^3 + 5s^4\cos\theta_1 + \tfrac{1}{2}s^5(7+35\cos^2\theta_1) + O(s^6)\}. \tag{2.2}$$

The region of low velocities, and hence high modified pressures, is therefore larger between the spheres at D than on the far side of A at C. If the resulting force on S_1 in the direction AO is F_1, the force coefficient

$$\frac{F_1}{\frac{1}{2}\pi a^2 \rho U^2} = 12s^4 + O(s^6). \tag{2.3}$$

Equation (2.3) is most easily derived from Basset's (1961) formulae for the kinetic energy, and Lagrange's equations. They also show that if AO is not quite parallel to the velocities of the spheres (assumed equal to each other), the force is still in the direction AO and still given by (2.3). This fact is used in §4 below. The sphere S_1 is thus repelled from S_2 by a force varying as s^4 for small s. Either symmetry or Newton's third law with d'Alembert's 'paradox' shows that S_1 repels S_2 with an equal and opposite force. If more than two spheres are moving with equal speeds in a straight line, it is easy to see by superposing velocities in (2.2) that the repulsion between each pair is given by (2.3) and that the error in the resultant force on any sphere is at most $O(s_n^6)$, where s_n is the value of s for its nearest neighbour.

3. The boundary layers

Denote the nth bubble in the line by S_n, starting at the upstream end. We ignore the terms which are $O(s^3)$ in (2.2), and let the θ_n component of velocity in the boundary layer be $\bar{q}_\theta + q_{\theta n}$, where

$$q_{\theta n} = UR^{-\frac{1}{2}} f(x,z)\operatorname{cosec}\theta_n, \tag{3.1}$$

$$x = n - 1 + \tfrac{1}{4}(2 - 3\cos\theta_n + \cos^3\theta_n), \tag{3.2}$$

$$z = \frac{3R^{\frac{1}{2}}(r_n - a)\sin^2\theta_n}{8a}, \tag{3.3}$$

and $n-1 \leqslant x \leqslant n$ on the nth bubble.

The argument of Harper & Moore (1968) adapted to this notation shows that $f(x,z)$ obeys the diffusion equation

$$\partial^2 f/\partial z^2 = 4(\partial f/\partial x), \tag{3.4}$$

in $n-1 < x < n$, $0 \leqslant z < \infty$, in the limit as $R \to \infty$, with initial condition

$$f(0,z) = 0, \tag{3.5}$$

continuity condition between the rear of one bubble and the front of the next

$$\lim_{x \to n-} f(x,z) = \lim_{x \to n+} f(x,z), \tag{3.6}$$

and bubble surface condition

$$\partial f/\partial z = 8 \quad \text{at} \quad z = 0, \tag{3.7}$$

provided that the action of viscosity in the wake between bubbles can be ignored, i.e. provided that the whole line of bubbles is short compared with a length $aR^{\frac{1}{2}}$ (see Moore 1963).

The solution for $f(x,z)$ is then obtained, as in Carslaw & Jaeger (1959, § 2·5), as

$$f(x,z) = -8x^{\frac{1}{2}}\,\mathrm{ierfc}\,(zx^{-\frac{1}{2}}) = -8x^{\frac{1}{2}}\int_{zx^{-\frac{1}{2}}}^{\infty} \mathrm{erfc}\; t\,dt, \tag{3.8}$$

for all $x > 0$. In each bubble the diffusion of $f(x,z)$ therefore appears to continue just where it left off in the one before. Such a simple situation could not occur with drops, where the internal viscosity forces some redistribution of the function $f(x,z)$ in the front stagnation region.

The viscous drag coefficients C_D may now be found by the method of Moore (1963) to be given by

$$\begin{aligned} R^{\frac{1}{2}}(RC_D - 48) = K_n &= \frac{3}{8}\int_0^{\infty} \{f(n,z)^2 - f(n-1)^2\}\,dz + \int_{-1}^{1} f(n-1+x,0)\,d\mu \\ &\quad + \frac{1}{16}\int_0^{\infty}\int_{-1}^{1}\left(\frac{\partial f}{\partial z}\right)^2 d\mu\, dz + O(R^{-\frac{1}{3}}) + O(R^{-\frac{1}{2}}s^{-1}) \\ &= 8(\sqrt{2}-1)\,\pi^{-\frac{1}{2}}\{n^{\frac{3}{2}} - (n-1)^{\frac{3}{2}}\} - 2\sqrt{(2/\pi)}\int_{-1}^{1}(4n-2-3\mu+\mu^3)^{\frac{1}{2}}\,d\mu \\ &\quad + O(R^{-\frac{1}{3}}) + O(R^{-\frac{1}{2}}s^{-1}), \end{aligned} \tag{3.9}$$

where $\mu = \cos\theta_n$. Hence $K_1 = -2{\cdot}211$, $K_2 = -4{\cdot}345$, $K_n > K_{n+1}$. Apart from the notation, (3.9) differs from Moore's result only in the first integral, which derives from viscous dissipation in the far wake, and in an additional term in the error, from viscous diffusion of (vorticity$/r_n \sin\theta_n$) between bubbles. To obtain (3.9), consider first an isolated bubble ($n = 1$). The right-hand side reduces to Moore's value of 2·211, to four figures. Bringing in a second bubble alters the velocity field around the first only by $O(s^3)$ of itself, and hence alters the viscous drag on it by $O(R^{-1}s^3)$, which we neglect. Equation (3.9) for $n = 2$ now follows because the total viscous dissipation in the flow field must, in steady motion, be equal to the sum of the powers of the drag forces on the two bubbles. Similarly, a third bubble changes the drag forces on the first two only by negligibly small amounts, and the viscous force on it must be the difference between the total drag forces for two and three bubbles, and so on. The term $O(R^{-\frac{1}{2}}s^{-1})$ in the drag equation (3.9) follows from Moore's theory of the wake, applied to the part of it between bubbles.

4. Lateral stability

A line of bubbles will remain vertical only if it is stable to small lateral disturbances. We show that if the second bubble S_2 is given a horizontal displacement ϵa, the horizontal force on it tends to increase ϵ, at least in the limit $\epsilon \to 0$, when higher powers than the first can be neglected. We take $\epsilon \ll R^{-\frac{1}{4}}$, so that the fluid from the wake of S_1 washes over the whole surface of S_2.

Although the motion is no longer axially symmetric, the irrotational flow still is, and so is the boundary layer around S_1, with error $O(\epsilon s^4)$, which we neglect. We may therefore use (3.8) to write $f = -8\,\mathrm{ierfc}\, z_1$ in the rear stagnation region of S_1, where we rename z as z_1, and where the irrotational stream function $\overline{\psi}_1$ is given by

$$\overline{\psi}_1 = 4Ua^2R^{-\frac{1}{2}}z_1. \tag{4.1}$$

In the wake between the bubbles

$$\overline{\psi}_1 = \tfrac{1}{2}Um_1^2, \tag{4.2}$$

where m_1 denotes distance from the vertical line $\chi_2 = 0$, $\theta_1 = 0$ or π, through the centre of S_1. With m_2 similarly defined from the line $\chi_1 = \pi$, $\theta_2 = 0$ or π, and $\overline{\psi}_2$ the inviscid stream function vanishing on S_2, we have $\overline{\psi}_2 = \frac{1}{2}Um_2^2$ in the wake between the bubbles, and to order ϵ

$$m_1^2 = m_2^2 - 2m_2\epsilon a\cos\chi_2, \quad \epsilon a\sin\chi_2 = m_1\sin(\chi_1-\chi_2). \tag{4.3}$$

In cylindrical polar co-ordinates (m_1, h, χ_1) the vorticity components in the wake of S_1 are $(0, 0, m_1B(\overline{\psi}_1))$, where $B(\overline{\psi}_1) = B(4Ua^2R^{-\frac{1}{2}}z_1) = 3U\,\mathrm{erfc}\,(z_1)/a^2$, and h measures distance vertically downwards. In co-ordinates (m_2, h, χ_2) the vorticity vector is therefore

$$\begin{aligned}(-m_1B(\overline{\psi}_1)\sin(\chi_1-\chi_2), 0, m_1B(\overline{\psi}_1)\cos(\chi_1-\chi_2))\\ = (-\epsilon a\sin\chi_2 B(\overline{\psi}_1), 0, Q(\overline{\psi}_1)),\end{aligned} \tag{4.4}$$

where $Q(\psi) = (2\psi/U)^{\frac{1}{2}}B(\psi)$, if we make use of (4.2) and (4.3).

Developments in the front stagnation region of the second bubble S_2 are best described in Boussinesq's orthogonal co-ordinate system (ψ, ϕ, χ), where ψ and ϕ are the stream function and velocity potential for the irrotational flow past S_2 in the absence of S_1, and $\chi = \chi_2$. In the wake between S_1 and S_2 and not close to either, this co-ordinate system is very nearly the same as the cylindrical polar one (m_2, h, χ_2), and (4.4) gives the vorticity components to a degree of accuracy sufficient for what follows. In the front stagnation region of S_2 the flow is an effectively inviscid small perturbation of that described by ψ and ϕ. We may therefore use Helmholtz's theorem that vorticity is convected like line elements of fluid, and obtain, correct to order ϵ,

$$\Omega_\psi = m_2V\omega_\psi = -\epsilon Ua\sin\chi Q(\psi), \tag{4.5}$$

$$\Omega_\phi = \frac{1}{V}\omega_\phi = -\epsilon Ua\sin\chi Q(\psi)\int\frac{e_{\psi\phi}}{m_2V^4}d\phi, \tag{4.6}$$

$$\Omega_\chi = \frac{1}{m_2}\omega_\chi = B(\psi) - \epsilon Ua\cos\chi\frac{dQ(\psi)}{d\psi}, \tag{4.7}$$

where $e_{\psi\phi}$ is the $\psi\phi$ rate-of-strain component in the actual flow, V is the speed of the irrotational flow (ψ, ϕ), and the integration in (4.6) is carried out along a streamline of that flow, i.e. $\psi =$ const., $\chi =$ const. The term $B(\psi)$ in (4.7) gives the starting value for the axially symmetric part of the motion, which was described in §3. The remaining terms in (4.5), (4.6) and (4.7) are all proportional to ϵ, and to begin the boundary-layer analysis we need only find the value of the integral in (4.6) right through the stagnation region. This turns out to come principally from the irrotational flow and to be given by

$$\int_{\substack{\text{stagnation}\\ \text{region}}} \frac{e_{\psi\phi}}{m_2 V^4}\, d\phi = \frac{a}{3U\psi}, \tag{4.8}$$

corrections induced by the vorticity being $O(R^{-\frac{1}{6}})$ smaller (Moore 1963).

Equations (4.5) to (4.8) can now be used to obtain the starting values of vorticity components, or more conveniently the Ω's, in the second bubble's boundary layer. The vorticity equations now include viscous terms, and it is elementary but time-consuming to show that Ω_ψ, Ω_ϕ and Ω_χ all obey the same one-dimensional diffusion equation (3.4) as $f(x, z)$. The boundary conditions to be applied at the surface are that tangential shear and normal velocity components vanish, and after a little reduction

$$\Omega_\phi = \Omega_\chi = \frac{\partial}{\partial z}(\Omega_\psi) = 0. \tag{4.9}$$

Calculation of the Ω's in the boundary layer now proceeds in the usual way (Carslaw & Jaeger 1959, §2.2). We require the lift coefficient C_L on the bubble, which is most easily found by following the variation of the vorticity through the rear stagnation region to the wake and working out the total line doublet strength of the trailing vortices (Batchelor 1967, p. 377). Thus, to order ϵ,

$$C_L = \epsilon R^{-\frac{1}{2}}(I_1 + I_2) + O(\epsilon s^5) + O(\epsilon R^{-\frac{2}{3}}) + O(\epsilon s^4 R^{-\frac{1}{2}}),$$

where

$$I_1 = \frac{16}{\pi^{\frac{1}{2}}} \int_0^\infty \int_0^\infty \left(\frac{z}{z'}\right)^{\frac{1}{2}} \operatorname{erfc}(z') \{e^{-(z-z')^2} - e^{-(z+z')^2}\}\, dz\, dz',$$

$$I_2 = \frac{16}{\pi^{\frac{1}{2}}} \int_0^\infty \int_0^\infty \left(\frac{z'}{z}\right)^{\frac{1}{2}} \operatorname{erfc}(z') \{e^{-(z-z')^2} + e^{-(z+z')^2}\}\, dz\, dz',$$

and after numerical integrations

$$C_L = 14{\cdot}4\epsilon R^{-\frac{1}{2}} + O(\epsilon s^5) + O(\epsilon R^{-\frac{2}{3}}) + O(\epsilon s^4 R^{-\frac{1}{2}}), \tag{4.10}$$

where the three error terms come respectively from pressure forces (§2), viscous corrections to $e_{\psi\phi}$ and asymmetry corrections to $e_{\psi\phi}$. The lack of axial symmetry also leads to extra terms in equations (4.5) to (4.7), but these have still smaller effects on (4.10).

The lift force is in a direction tending to increase ϵ, and so in a pure liquid a vertical line of two bubbles is unstable. Clearly the same would hold for more bubbles: with the first n bubbles in line and the $(n+1)$th out a distance ϵa, the factors erfc (z') in I_1 and I_2 would be replaced by erfc $(z'/n^{\frac{1}{2}})$ on the theory of §3, or some more complicated positive function of z' on taking diffusion of vorticity between bubbles into account, but I_1 and I_2 would both remain positive.

I wish to thank Mr D. Laws for giving me a copy of his computer procedure for numerical integration, and the Applied Mathematics Division of the New Zealand Department of Scientific and Industrial Research and the Information Science Department of the Victoria University of Wellington for the opportunity to use it.

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